Wiener Odd and Even Indices on BC-Trees

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***Abstract*—Corresponding to the concepts of Wiener index and distance of the vertex, in this paper, we present the concepts of Wiener odd (even) index of G as sum of the distances between all pairs of vertices of G satisfying the distances are all odd (even) and denote them $W_{odd}(G)$ and $W_{odd}(G)$ respectively. Based on the concepts of the two indices, we prove theoretically that Wiener odd index is not more than its even index for general BC-Trees. Closed formulae of the two indices are also provided for path BC-tree, star, k-extending star tree and caterpillar BC-tree. Meanwhile, the extreme values of $W_{odd}(T)$ of n vertices BC-trees are characterized as well.**

***Keywords*-Wiener odd (even) index; odd (even) distance of the vertex; BC-tree; k-extending star; caterpillar BC-tree;**

I. Introduction

All graphs $G = (V(G), E(G))$ in this paper will be a finite, undirected and connected, $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. Let $d_G(u,v)$ (or simply $d(u,v)$ when no confusion arises) denote the distance between the vertices u and v in G. For a vertex v of G, define the distance of the vertex as the sum of distances from v to all other vertices.

$$g_G(v) = \sum_{u \in V(G)} d_G(v,u),$$

define the odd(even) distance of the vertex v as the sum of distances from v to all other vertices of G satisfying the distances are all odd(even). i.e.

$$g_{odd,G}(v) = \sum_{u \in V(G) \wedge d_G(v,u) \equiv 1 (mod\ 2)} d_G(v,u),$$

and

$$g_{even,G}(v) = \sum_{u \in V(G) \wedge d_G(v,u) \equiv 0 (mod\ 2)} d_G(v,u).$$

Let $W(G) = \frac{1}{2} \sum\limits_{v \in V(G)} g_G(v)$ denote the Wiener index of G, which is the sum of distances for all unordered pairs of vertices. Let $W_{odd}(G) = \frac{1}{2} \sum\limits_{v \in V(G)} g_{odd,G}(v) (W_{even}(G) = \frac{1}{2} \sum\limits_{v \in V(G)} g_{even,G}(v))$ denote the Wiener odd(even) index of G, which is the sum of distances between all unordered pairs of vertices satisfying the distances are all odd(even). Obviously, we have $g_G(v) = g_{odd,G}(v) + g_{even,G}(v)$, $W(G) = W_{odd}(G) + W_{even}(G)$

A tree $T = (V, E)$ is a connected, acyclic graph. A vertex of degree one will be called a pendent vertex or leaf of T. A tree on n vertices has at least 2 and at most $n-1$ pendent vertices. The (unique) n-vertex trees with 2 and $n-1$ pendent vertices are called the path and star, respectively, and are denoted by P_n and $K_{1,n-1}$, respectively.

A tree is called a BC-tree (the block-cutpoint-tree or the bicolorable tree) if the distance between any two leaves is even see Harary and Plummer [1], [2]. BC-tree have many special properties and applications in graph theory, chemistry, computer and brain science, see [3], [4], [5], [6], [7].

Define k-extending star tree the tree constructed by adding $n-1$ paths with length $k-1$ to each of the $n-1$ pendant vertices of the star $K_{1,n-1}$ denoted by $K^k_{1,n-1}$, the center vertex is denoted as c, for any vertex $u \in V(K^k_{1,n-1})$, Let the distance $d_{K^k_{1,n-1}}(u,c)$ between u and c the height of u. By this definition, $K^k_{1,n-1}$ is the star $K_{1,n-1}$ when $k = 1$.

A caterpillar tree is a tree, which has a path, such that every vertex not on the path is adjacent to some vertex on the path. A caterpillar BC-tree is both a caterpillar tree and a BC-tree. P_n is also a BC-tree, when n is odd. For standard notations in graph theory, [8], [9] may be consulted.

The rest of the paper is organized as follows. Section II describes related works on Wiener index and BC-trees. In Section III, we study the relationship between Wiener odd index $W_{odd}(G)$ and Wiener even index $W_{odd}(G)$ of general BC-tree, path BC-tree, star, k-extending star tree, caterpillar BC-tree, and we also give out the maximum and minimum value of $W_{odd}(T)$ and the extremal trees attaining these values as well. Finally the conclusions and some open problems are presented in Section IV.

II. Related works

The Wiener index was first developed by Wiener [10] in 1947. And the graphical invariant $W(G)$ has been studied in [11], [12], [13], [14] under different names such as distance, transmission, total status and sum of all distances. This concept has been one of the most widely used descriptors in quantitative structure activity relationships, as the Wiener index has been shown to have a strong correlation with the chemical properties of a chemical compound [15]. The Wiener index and the average distance rank among those graph-theoretical parameters that are of most interest in other fields. Dobrynin [11] provided a very comprehensive survey about Wiener index.

Chemical structures of organic compounds are characterized numerically by a variety of structural descriptors, one of the earliest and most widely used being the Wiener index W, derived from the interatomic distances in a molecular graph. Extensive use of such structural descriptors or topological indices has been made in drug design, screening of chemical databases, and similarity and diversity assessment. A new set of topological indices is introduced representing a partitioning of the Wiener index based on counts of even and odd molecular graph distances by Ivanciuc [16]. These new indices are further generalized by weighting exponents which can be optimized during the quantitative structure-activity/-property relationship (QSAR/QSPR) modeling process. In [16], Ivanciuc also tested these novel topological indices in QSPR models for the boiling temperature, molar heat capacity, standard Gibbs energy of formation, vaporization enthalpy, refractive index, and density of alkanes, and Ivanciuc [16] conluded that in many cases, the even/odd distance indices proposed here give notably improved correlations.

Namely, like the Wiener index, the Wiener odd and even index of graph may also be meaningful topological index, which can be used in mathematics, chemistry, bioinformatics and brain science. Hence, the study of Wiener odd and even index of graph is of great significance.

III. Wiener odd (even) index on BC-trees

A. *Wiener odd (even) index on general BC-trees*

Since the conception of the block-cutpoint-tree (i.e. BC-tree) was proposed, it has been drawing researchers' attention and concern worldwide, not only in the field of mathematics and computer science, but also in chemistry and brain science. See Barefoot [5], Mkrtchyan [17], Misiołek and Chen [18], Paton [19], Gagarin and Labelle [20], Nakayama and Fujiwara [21], Yang and Wang [22]

It is well known that the Wiener index among trees on n vertices is minimized by the star $K_{1,n-1}$ and is maximized by the n-vertex path P_n, see Entringer et al. [23], or Lovász [24].

Since BC-tree is a tree with special nature, next, we will examine the the Wiener odd (even) index on BC-tree.

Lemma 1: [23] The Wiener index of n-vertex path P_n is $(n-1)^2$, more than any other tree on n vertices. The Wiener index of star $K_{1,n-1}$ is $(n^3-n)/6$, fewer than any other tree on n vertices.

Theorem 3.1: Let T be a BC-tree on n vertices, then $W_{odd}(T) \le W_{even}(T)$.

Proof: For any BC-tree T, we can always find a vertex v, such that each two trees after splitting at v are also BC-trees, and one of them is a star. For illustration see Fig. 1. Let T'' be a star on $p+2(p\ge 1)$ vertices, for brevity, set

$$\begin{aligned}
&S_1=\{w|w\in V(T')\backslash v\wedge d_{T'}(w,v)\equiv 0(mod\ 2)\},\\
&S_2=\{w|w\in V(T')\wedge d_{T'}(w,v)\equiv 1(mod\ 2)\},\\
&R_1=\{w|w\in V(T'')\backslash v\wedge d_{T''}(w,v)\equiv 0(mod\ 2)\},\\
&R_2=\{w|w\in V(T'')\wedge d_{T''}(w,v)\equiv 1(mod\ 2)\},\\
&\quad N_1=|S_1|,\quad N_2=|S_2|,\quad M_1=|R_1|,\quad M_2=|R_2|,\\
&g_{even,T'}(v)=\sum_{w\in S_1} d_{T'}(w,v),\ g_{odd,T'}(v)=\sum_{w\in S_2} d_{T'}(w,v)\\
&g_{even,T''}(v)=\sum_{w\in R_1} d_{T''}(w,v),\ g_{odd,T''}(v)=\sum_{w\in R_2} d_{T''}(w,v)
\end{aligned}$$

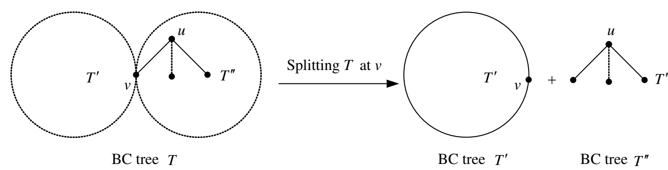


Figure 1. Split of BC-tree T.

Since T' is a BC-tree, By the definition of BC-tree, it is not difficult to obtain

$$N_1\ge N_2, g_{odd,T'}(v)\le g_{even,T'}(v). \tag{1}$$

It is clear that the theorem holds for $n=3,4,5$, suppose the theorem holds for BC-trees of orders $<n$, then, we categorise the unordered pairs of vertices of T satisfying the distances are all odd as follow:

$(1)\{(p,q)|p\in V(T''), q\in V(T'')\wedge d_{T''}(p,q)\equiv 1(mod\ 2)\}$;
$(2)\{(p,q)|p\in V(T'), q\in V(T')\wedge d_{T'}(p,q)\equiv 1(mod\ 2)\}$;
$(3)\{(p,q)|p\in R_1, q\in S_2\}$;
$(4)\{(p,q)|p\in R_2, q\in S_1\}$.

Similarly, the unordered pairs of vertices of T satisfying the distances are all even are as follow:

$(5)\{(p,q)|p\in V(T^{''}), q\in V(T^{''})\wedge d_{T''}(p,q)\equiv 0(mod\ 2)\}$;
$(6)\{(p,q)|p\in V(T^{'}), q\in V(T^{'})\wedge d_{T'}(p,q)\equiv 0(mod\ 2)\}$;
$(7)\{(p,q)|p\in R_1, q\in S_1\}$;
$(8)\{(p,q)|p\in R_2, q\in S_2\}$.

With the above notations, it is easy to know that $M_1=p, M_2=1$. By analyzing, the sum of the distances of the unordered pairs of vertices of cases (3)+(4) and cases (7)+(8) is $p\times g_{odd,T'}(v)+2pN_2+g_{even,T'}(v)+N_1$ and $p\times g_{even,T'}(v)+2pN_1+g_{odd,T'}(v)+N_2$ respectively.

Obviously, cases (1) and (2) are $W_{odd}(T')$ and $W_{odd}(T'')$; cases (5) and (6) are $W_{even}(T')$ and $W_{even}(T'')$. Therefore,

$$W_{odd}(T) - W_{even}(T) =$$
$$W_{odd}(T') - W_{even}(T') + W_{odd}(T'') - W_{even}(T'')$$
$$+ (p-1)(g_{odd,T'}(v) - g_{even,T'}(v)) + (2p-1)(N_2 - N_1) \quad (2)$$

By induction, $W_{odd}(T') \leq W_{even}(T')$, $W_{odd}(T'') \leq W_{even}(T'')$, since $p \geq 1$, combining (1), we obtain $W_{odd}(T) \leq W_{even}(T)$. The theorem thus holds. ■

Theorem 3.2: The Wiener odd index of star $K_{1,n-1}$ is $n-1$, fewer than any other BC-trees on n vertices and the maximum value of Wiener odd index is $(n^3 - n)/12$.

Proof: Let T be an arbitrary n vertices BC-tree, It's easy to see that $W_{odd}(T) \geq n-1$, and it is not hard to observe that $W_{odd}(K_{1,n-1}) = n-1$, therefore, star $K_{1,n-1}$ minimizes the Wiener odd index. Next, we prove any other BC-tree on n vertices can not attain this value.

Suppose l is a leaf of T, v is the neighbor vertex of l, if the distances between all other vertices and v is 2, then they are neighbors of v, otherwise, there will exist a vertex w, such that $d_T(l, w) = 3$, then $W_{odd}(T) > n-1$. Hence T is star.

By Theorem 3.1 and Lemma 1, we have

$$W_{odd}(T) \leq W(T)/2 = (n^3 - n)/12.$$

By simple calculation, we have $W_{odd}(P_{n-1}) = W_{even}(P_{n-1}) = (n^3 - n)/12$ for n is odd, but it isn't the unique BC-tree that can attain this maximum value. ■

Székely and Wang [25] studied the problem of enumerating subtrees of trees. They proved the following results:

Lemma 2: (Székely and Wang [25]) The path P_n has $\binom{n+1}{2}$ subtrees, fewer than any other trees of n vertices. The star $K_{1,n-1}$ has $2^{n-1} + n - 1$ subtrees, more than any other trees of n vertices.

In [22], we studied the BC-subtrees of $K_{1,n-1}$ and P_n. And we have the following theorems.

Lemma 3: [22]The star $K_{1,n-1}$ has $2^{n-1} - n$ BC-subtrees, more than any other trees of n vertices.

Lemma 4: [22]The number of BC-subtrees of path P_n is

$$\eta_{BC}(P_n) = \begin{cases} n(n-2)/4 & n \equiv 0(mod\ 2), \\ (n-1)^2/4 & n \equiv 1(mod\ 2). \end{cases}$$

fewer than any other n-vertices tree.

We know that star $K_{1,n-1}$ is a BC-tree, and P_n is also a BC-tree when n is odd. By Theorem 3.2, Lemma 3, Lemma 4 we have $K_{1,n-1}$ minimizes the Wiener odd index and maximizes the BC-subtrees among BC-trees on n vertices; P_n maximizes the Wiener odd index and minimizes the BC-subtrees among BC-trees on n(n is odd) vertices.

We see here an amazing and not yet understood relationship between the Wiener odd index and the number of BC-subtrees which is "within certain classes of BC-trees of a fixed parameter, the smaller the number of BC-subtrees is, the bigger the Wiener odd index is". Unfortunately this relationship does not extend as expected. See Fig. 2. T_0 and $T_0^{'}$ are BC-trees on 12 vertices. Simple calculations show that $W_{odd}(T_0) = 69$, $W_{odd}(T_0') = 73$, $\eta_{BC}(T_0) = 183$, $\eta_{BC}(T_0') = 252$(where $\eta_{BC}(T_0)$ denotes the BC-subtrees number of T), we have $W_{odd}(T_0) < W_{odd}(T_0')$, though $\eta_{BC}(T_0) < \eta_{BC}(T_0')$.

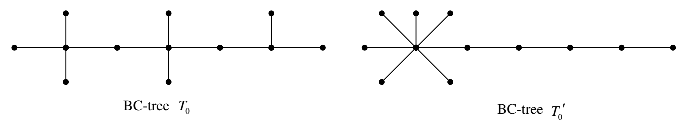


Figure 2. The counter-example caterpillar tree T

B. *Wiener odd (even) index on k-extending star trees*

Since k-extending star tree with 2 branches is a path BC-tree, by Theorem 3.2, its Wiener even index equals its Wiener odd index. Thus, we only consider $n \geq 4$.

Theorem 3.3: Let $K_{1,n-1}^k$ be a k-extending star tree with $n-1 (n \geq 4)$ branches, then, $W_{even}(K_{1,n-1}^k)$ equals $W_{odd}(K_{1,n-1}^k)$ (resp. $W_{odd}(K_{1,n-1}^k) + (k+1)(n-1)(n-3)/2$) for n is even(resp. odd).

Proof: We discuss it for k is even and odd respectively.

(1) For k is even, using the splitting method and denotations as in Theorem 3.1, and for the sake of convenience, for branch b_i $(i = 1, 2, \ldots, n-1)$, label the vertices of b_i as this: $b_i = v_{i0}v_{i1}v_{i2}\ldots v_{ik-2}v_{ik-1}v_{ik}$ where v_{i0} is the center vertex c and v_{ik} is the leaf vertex, then we split $K_{1,n-1}^k$ at vertex v_{ik-j} $(j = 2, 4, 6, \ldots, k-2, k)$ sequentially each time, T' and T'' are as in Fig. 3(a) after the first splitting.

Simple calculations show that $W_{odd}(T'') = W_{even}(T')$, $P = 1$, and $N_1 = N_2$, hence, by (2), $W_{odd}(T) - W_{even}(T) = W_{odd}(T') - W_{even}(T')$. Continue the splitting process until the whole branch b_i is deleted, denote the tree at this time as T_0, then

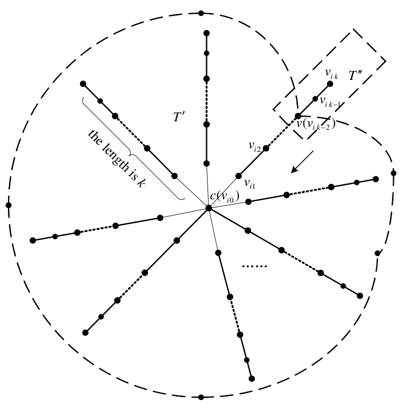
$$W_{odd}(T) - W_{even}(T) = W_{odd}(T_0) - W_{even}(T_0)$$

continue deleting this way until T becomes a k-extending star tree with 2 branches, we denote it as T_0', then we have

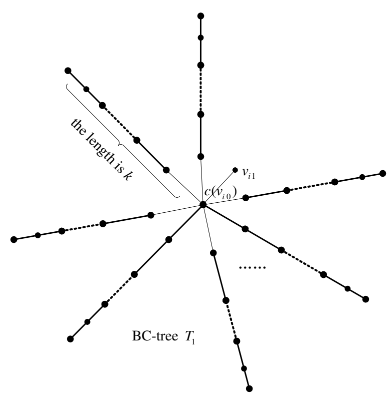
$$W_{odd}(T) - W_{even}(T) = W_{odd}(T_0') - W_{even}(T_0').$$

Since $W_{odd}(T_0') = W_{even}(T_0')$, therefore, $W_{odd}(T) = W_{even}(T)$.

(2) For k is odd, as the discussion for n is even, similarly, after the first splitting, we have $W_{odd}(T) - W_{even}(T) = W_{odd}(T') - W_{even}(T') - (n-3)$ by (2). Proceeding the splitting until T is splitted to a tree with only one leaf in height 1 and the other $n-2$ leaves are in height k, see Fig.



(a) The splitting illustration of $K_{1,n-1}^k$ (k is even)



(b) The splitting illustration of $K_{1,n-1}^k$ (k is odd)

Figure 3. The splitting illustration of $K_{1,n-1}^k$

3(b), and we denote at this time as T_1, then we have

$$W_{odd}(T) - W_{even}(T) = W_{odd}(T_1) - W_{even}(T_1) - (k-1)(n-3)/2.$$

Next, choose another branch, go on splitting, until T is split to a star $K_{1,n-1}$, then we have

$$W_{odd}(T) - W_{even}(T) = W_{odd}(K_{1,n-1}) - W_{even}(K_{1,n-1}) - (n-1)(k-1)(n-3)/2.$$

By Lemma 1 and Theorem 3.2, we have $W_{odd}(K_{1,n-1}) - W_{even}(K_{1,n-1}) = -(n-1)(n-3)$. Therefore

$$W_{even}(T) = W_{odd}(T) + (k+1)(n-1)(n-3)/2.$$

The theorem holds. ■

C. Wiener odd (even) index on caterpillar BC-trees

Caterpillars were first studied in a series of papers by Harary and Schwenk [26], [27]. And they have many characterizations and generalizations such as a chordal graph with exactly $n-k$ maximal cliques, each containing $k+1$ vertices which we call k-tree and lobster graph. Caterpillar trees have been used in chemical graph theory to represent the structure of benzenoid hydrocarbon molecules and received more and more attention by the researchers.

Combining the special property of BC-trees, next, we examine the caterpillar BC-tree.

Consider the caterpillar tree T with diameter l ($l >= 2$)(l is even by definition)in Fig. 4, x and y are leaves on the diameter. After the deletion of all the edges of $P_T(x, y)$ from T, some connected components will remain. Let T_i denote the star on $n_i + 1$ vertices with center vertex v_i, for $i = 1, 2, \ldots, l/2$ ($l \geq 2$).

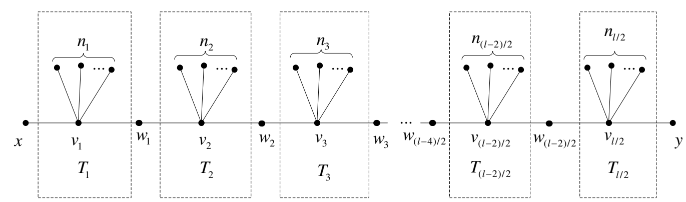


Figure 4. The caterpillar BC-tree T with diameter l.

Theorem 3.4: Let T be the caterpillar BC-tree described above. Then, $W_{even}(T) > W_{odd}(T)$ and

$$W_{even}(T) = W_{odd}(T) + \sum_{i=1}^{l/2} n_i(n_i+2) - \sum_{i=1}^{(l-2)/2} (2n_i+1) \sum_{j=i+1}^{l/2} n_j + \sum_{i=1}^{(l-2)/2} n_i(2\sum_{j=i+1}^{l/2} (j-i)(n_j+1) - (l-2i)^2/4).$$

Proof: We split the T as described in Theorem 3.1 at vertex w_i ($i = 1, 2, \ldots, (l-2)/2$) sequentially, then two parts obtained each time were denoted as T_i'' and T_i', where T_i'' is a star on $n_i + 3$ vertices and T_i' is a caterpillar BC-tree, and it is easy to observe that $T_i'' = K_{1,n_i+2}(i = 1, 2, \ldots, (l-2)/2)$, $T_i'(i = 1, 2, \ldots, (l-2)/2)$ is a caterpillar BC-tree. After splitting $(l-2)/2$ times, the caterpillar BC-tree T was splitted up.

For the sake of brevity, we give some denotations.

$$S_{i,1} = \{w | w \in V(T_i^{'}) \backslash w_i \wedge d_{T_i'}(w, w_i) \equiv 0(mod\ 2)\},$$
$$S_{i,2} = \{w | w \in V(T_i^{'}) \wedge d_{T_i'}(w, w_i) \equiv 1(mod\ 2)\},$$
$$N_{i,1} = |S_{i,1}|, \quad g_{even,T_i'}(w_i) = \sum_{w \in S_{i,1}} d_{T_i'}(w, w_i),$$
$$N_{i,2} = |S_{i,2}|, \quad g_{odd,T_i'}(w_i) = \sum_{w \in S_{i,2}} d_{T_i'}(w, w_i).$$

Meanwhile, we define $p_i = n_i + 1 (i = 1, 2, \ldots, \frac{(l-2)}{2})$.

After splitting T at w_1, Simple calculations show that

$$N_{1,1} = \sum_{j=2}^{l/2} n_j + l/2 - 1, N_{1,2} = l/2 - 1,$$

$$g_{even,T_1'}(w_1) = 2\sum_{j=2}^{l/2}(j-1)(n_j+1),$$

$$g_{odd,T_1'}(w_1) = (l-2)^2/4, \quad p_1 = n_1 + 1,$$

$$W_{odd}(T_1'') - W_{even}(T_1'') = -n_1(n_1+2).$$

By (2), we have

$$W_{odd}(T) - W_{even}(T) = W_{odd}(T_1') - W_{even}(T_1')$$
$$- n_1(n_1+2) + n_1((l-2)^2/4 - 2\sum_{j=2}^{l/2}(j-1)(n_j+1))$$
$$- (2n_1+1)\sum_{j=2}^{l/2} n_j.$$

Proceeding this way, after splitting $(l-2)/2$ times, we have

$$W_{odd}(T) - W_{even}(T) = W_{odd}(K_{1,n_{l/2}+2}) - W_{even}(K_{1,n_{l/2}+2})$$
$$+ \sum_{i=1}^{(l-2)/2} (W_{odd}(T_i'') - W_{even}(T_i''))$$
$$+ \sum_{i=1}^{(l-2)/2} (p_i - 1)(g_{odd,T_i'}(w_i) - g_{even,T_i'}(w_i))$$
$$+ \sum_{i=1}^{(l-2)/2} (2p_i - 1)(N_{i,2} - N_{i,1}).$$

i.e.

$$W_{odd}(T) - W_{even}(T) = W_{odd}(K_{1,n_{l/2}+2})$$
$$- W_{even}(K_{1,n_{l/2}+2}) - \sum_{i=1}^{(l-2)/2} n_i(n_i+2)$$
$$+ \sum_{i=1}^{(l-2)/2} n_i((l-2i)^2/4 - 2\sum_{j=i+1}^{l/2}(j-i)(n_j+1))$$
$$+ \sum_{i=1}^{(l-2)/2} (2n_i+1)\sum_{j=i+1}^{l/2} n_j.$$

Since $W_{odd}(K_{1,n_{l/2}+2}) - W_{even}(K_{1,n_{l/2}+2}) = -n_{l/2}(n_{l/2}+2)$, the theorem thus holds. ∎

Let T_φ, T_ϕ be caterpillar BC-trees with the structure as depicted in Fig. 4 with $n_i'(i = 1, 2, \ldots, \frac{l}{2})$, w_i' $(i = 1, 2, \ldots, \frac{l}{2} - 1)$ and n_i'' $(i = 1, 2, \ldots, \frac{l}{2})$, w_i'' $(i = 1, 2, \ldots, \frac{l}{2} - 1)$ respectively.

Theorem 3.5: Let T_φ, T_ϕ be described above satisfing the following condition: $n_i' + n'_{\frac{l}{2}+1-i} = n_i'' + n''_{\frac{l}{2}+1-i}$ $(i = 1, 2, \ldots, \lfloor \frac{l}{4} \rfloor)$, then, $W_{odd}(T_\varphi) = W_{odd}(T_\phi)$.

Proof: We split T_φ and T_ϕ at vertex w_i' $(i = 1, 2, \ldots, \frac{l}{2} - 1)$, w_i'' $(i = 1, 2, \ldots, \frac{l}{2} - 1)$ sequentially, then, after some algebraic operations, When $l = 4k$,

$$W_{odd}(T_\varphi) = \sum_{j=1}^{l/4} \frac{l^2 + 8j^2 + 4l - 4lj - 8j}{4}(n_j' - 2 + n'_{\frac{l}{2}+1-j} - 2) + l/2$$

$$W_{odd}(T_\phi) = \sum_{j=1}^{l/4} \frac{l^2 + 8j^2 + 4l - 4lj - 8j}{4}(n_j'' - 2 + n''_{\frac{l}{2}+1-j} - 2) + l/2$$

When $l = 4k + 2$,

$$W_{odd}(T_\varphi) = \sum_{j=1}^{(l-2)/4} \frac{l^2 + 8j^2 + 4l - 4lj - 8j}{4}(n_j' - 2 + n'_{\frac{l}{2}+1-j} - 2)$$
$$+ l/2 + \frac{l^2 + 4l - 4}{8}(n'_{(l+2)/4} - 2)$$

$$W_{odd}(T_\phi) = \sum_{j=1}^{(l-2)/4} \frac{l^2 + 8j^2 + 4l - 4lj - 8j}{4}(n_j'' - 2 + n''_{\frac{l}{2}+1-j} - 2)$$
$$+ l/2 + \frac{l^2 + 4l - 4}{8}(n''_{(l+2)/4} - 2)$$

In either case, we know that if $n_i' + n'_{\frac{l}{2}+1-i} = n_i'' + n''_{\frac{l}{2}+1-i}$ $(i = 1, 2, \ldots, \lfloor \frac{l}{4} \rfloor)$, then, $W_{odd}(T_\varphi) = W_{odd}(T_\phi)$. ∎

IV. Conclusions and Further Works

In this paper, we presented the concepts of odd (even) distance of the vertex v as the sum of distances from v to all other vertices of G satisfying the distances are all odd (even). We illustrated Wiener odd (even) index of G as the sum of the distances between all pairs of vertices of G satisfying the distances are all odd (even), which are denoted as $W_{odd}(G)$ $(W_{even}(G))$ respectively. And we proved that the Wiener odd index is not more than its even index for general BC-Trees. For k-extending star trees and caterpillar BC-trees, we gave out the equalities of their Wiener even index and odd index. We also gave out the maximum value $(n^3 - n)/12$ and minimum value $n - 1$ of $W_{odd}(T)$ on n vertices BC-trees and the extremal BC-trees attaining these values as well.

As the widely researched Wiener index, we can also do some researches on odd (even) distance of vertex and

Wiener odd (even) index on special trees and general trees correspondingly and comprehensively; Obviously, Wiener odd and even index can also be regarded as topological index. Hence another interesting direction is to explore the role of these index in graph theory, chemistry and brain networks.

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REFERENCES

[1] F. Harary and G. Prins, "The block-cutpoint-tree of a graph," *Publicationes Mathematicae - Debrecen*, vol. 13, pp. 103–107, 1966.

[2] F. Harary and M. Plummer, "On the core of a graph," *Proceedings London Mathematical Society*, vol. 17, pp. 249–257, 1967.

[3] R. Johannesson and K. Zigangirov, "On the distribution of the number of computations in any finite number of subtrees for the stack algorithm (corresp.)," *IEEE Transactions on Information Theory*, vol. 31, no. 1, pp. 100–102, 1985.

[4] J. Barnard, "A comparison of different approaches to markush structure handling," *Journal of Chemical Information and Computer Sciences*, vol. 31, no. 1, pp. 64–68, 1991.

[5] C. Barefoot, "Block-cutvertex trees and block-cutvertex partitions," *Discrete Mathematics*, vol. 256, no. 1, pp. 35–54, 2002.

[6] A. Symeonidis and I. Tollis, "Visualization of biological information with circular drawings," *Biological and Medical Data Analysis, Lecture Notes in Computer Science*, vol. 3337, pp. 468–478, 2004.

[7] F. De Vico Fallani, L. Astolfi, F. Cincotti, D. Mattia, A. Tocci, S. Salinari, M. Marciani, H. Witte, A. Colosimo, and F. Babiloni, "Brain network analysis from high-resolution eeg recordings by the application of theoretical graph indexes," *IEEE Transactions on Neural Systems and Rehabilitation Engineering*, vol. 16, no. 5, pp. 442–452, 2008.

[8] J. A. Bondy, *Graph Theory with Applications*. The Macmillan Press Ltd, 1979.

[9] F. Harary, *Graph Theory*. Addison-Wesley, Reading, MA, 1969.

[10] H. Wiener, "Structural determination of paraffin boiling points," *Journal of the American Chemical Society*, vol. 69, no. 1, pp. 17–20, 1947.

[11] A. Dobrynin, R. Entringer, and I. Gutman, "Wiener index of trees: theory and applications," *Acta Applicandae Mathematicae*, vol. 66, no. 3, pp. 211–249, 2001.

[12] M. Fischermann, A. Hoffmann, D. Rautenbach, L. Székely, and L. Volkmann, "Wiener index versus maximum degree in trees," *Discrete Applied Mathematics*, vol. 122, no. 1, pp. 127–137, 2002.

[13] F. Jelen and E. Triesch, "Superdominance order and distance of trees with bounded maximum degree," *Discrete Applied Mathematics*, vol. 125, no. 2, pp. 225–233, 2003.

[14] S. Klavžar and I. Gutman, "Wiener number of vertex-weighted graphs and a chemical application," *Discrete Applied Mathematics*, vol. 80, no. 1, pp. 73–81, 1997.

[15] I. Gutman and T. Kortvelyesi, "Wiener indices and molecular surfaces," *Zeitschrift für Naturforschung. A, A Journal of physical sciences*, vol. 50, no. 7, pp. 669–671, 1995.

[16] O. Ivanciuc, T. Ivanciuc, D. Klein, W. Seitz, and A. Balaban, "Wiener index extension by counting even/odd graph distances," *Journal of Chemical Information and Computer Sciences*, vol. 41, no. 3, pp. 536–549, 2001.

[17] V. Mkrtchyan, "On trees with a maximum proper partial 0-1 coloring containing a maximum matching," *Discrete Mathematics*, vol. 306, pp. 456–459, 2006.

[18] E. Misiolek and D. Z. Chen, "Two flow network simplification algorithms," *Information Processing Letters*, vol. 97, pp. 197–202, 2006.

[19] K. Paton, "An algorithm for the blocks and cutnodes of a graph," *Communications of The ACM*, vol. 14, pp. 468–475, 1971.

[20] A. Gagarin and G. Labelle, "Two-connected graphs with prescribed three-connected components," *Advances in Applied Mathematics*, vol. 43, no. 1, pp. 46–74, 2009.

[21] T. Nakayama and Y. Fujiwara, "BCT representation of chemical structures," *Journal of Chemical Information and Computer Sciences*, vol. 20, no. 1, pp. 23–28, 1980.

[22] Y. Yang, D. Wang, H. Wang, and H. Liu, "On BC-trees and BC-subtrees," *arXiv preprint arXiv:1305.4711*, 2013.

[23] R. Entringer, D. Jackson, and D. Snyder, "Distance in graphs," *Czechoslovak Mathematical Journal*, vol. 26, no. 2, pp. 283–296, 1976.

[24] L. Lovász, *Combinatorial problems and exercises*. American Mathematical Society, 1993.

[25] L. Székely and H. Wang, "On subtrees of trees," *Advances in Applied Mathematics*, vol. 34, no. 1, pp. 138–155, 2005.

[26] F. Harary and A. Schwenk, "The number of caterpillars," *Discrete Mathematics*, vol. 6, no. 4, pp. 359–365, 1973.

[27] S. El-Basil, "Applications of caterpillar trees in chemistry and physics," *Journal of Mathematical Chemistry*, vol. 1, no. 2, pp. 153–174, 1987.